Sign test

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Statistics for Business

Sign test

Definition

Sign test is a non-parametric test, a special case for the binomial test with $p = 1/2$, with these applications:

- for a simple sample, testing is the population has a given value for the median; e.g., given the sample $3.1, 2.4, 5.6, 6.7, 4.2, 3.8$, we may test by the sign test if the sample is compatible with a population with $Me = 3.5$.
- for paired samples, testing is the difference of medians for both populations us 0, that is, if one population takes bigger values than the other one. E.g., after applying a given fertilizer, to test is a plant gives better returns.
- Setting confidence intervals for the median.

- Non-parametric tests are tests that don't set a given distribution for the population, so they are more flexible.
- By means of a binomial test, we decide if a given p probability is compatible with data.

Test about the median (I)

- The median is the value that leaves below it 50% of data, that is, a 0.5 probability.
- Hence, in a population the probability for one data being below the median is 0.5.
- To perform the test, variable must be continuous, that is, it must take many different values.
- Under the null hypothesis, we set a given value for the median: $H_0 : Me = m$. Alternatively, $H_a : Me \neq m$.

Test about the median (II)

- We assign the $-$ sign to the data below $Me = m$; to the data above $Me = m$, we assign the $+$ sign.
- We count $-$ and $+$ data: r^- and r^+.
- This test is two-sided: we reject the null hypothesis, $H_0 : Me = m$, when r^- and r^+ values are very big and very small. In order to accept H_0, r^- and r^+ should be close to the number of data divided by 2.
- But, in order to perform the test in a standard way, we take always the smallest value between r^- and r^+. We name this value r.
- As is the smallest value, we reject $H_0 : Me = m$ when r is small enough.

Test about the median (III)

- For small samples, we look for the r^* critical value into the tables, for different values of the α significance level.
- We reject the null hypothesis, that is, the value given for the median, when we have $r \leq r^*$.
- As the variable is continuous, in theory there is no data equal to the median value, but if it were, we would remove it.

Test about the median (IV): calculating p value

- The probability for one data of being below of above the median in the null hypothesis is 0.5.
- For n data, the number of data below or above the median is distributed $B(n, 0.5)$.
- For n data, the smallest number of signs will be x when the number of data below median are x or data above median is x.
- And the probability for that event is:

$$\begin{aligned} P[min = x] &= P[above\ Me = x] + P[below\ Me = x] \\ &= 0.5^x 0.5^{n-x} \frac{n!}{x!(n-x)!} + 0.5^x 0.5^{n-x} \frac{n!}{x!(n-x)!} \\ &= 2 \times 0.5^n \frac{n!}{x!(n-x)!} \end{aligned}$$

- The p-value is the probability for the evidence or something stranger. As we took the minimum value, "the strange thing" is on the lower side. So:

$$p = P[X \leq x] = 2 \sum_{i=1}^{x} 0.5^n \frac{n!}{i!(n-i)!} = 2 \times 0.5^n \sum_{i=1}^{x} \frac{n!}{i!(n-i)!}$$

Test about the median (V): example

- Data: 6.14-6.45-8.33-11.05-5.67-5.22-7.52-10.08-12.34
- We want to test if the median in the population is 7: $H_0 : Me = 7$.
  We set $\alpha = 10\%$.
- We set the signs: $-, -, +, +, -, -, +, +, +$.
- $r^- = 4; r^+ = 5$. We take the smallest one: $r = 4$.
- We look for the critical value into the table: $r^* = 1$.
- As r is above that value, we accept the null hypothesis, that is, that median is 7.
- The p-value is the probability of having 4 data below the median (and not above the median):

$$p = P[X \leq 4] = P[X = 0] + P[X = 1] + P[X = 2] + P[X = 3] + P[X = 4]$$
$$= 0.5^0 \cdot 0.5^9 \cdot \frac{9!}{0!9!} + 0.5^1 \cdot 0.5^8 \cdot \frac{9!}{1!8!} + 0.5^2 \cdot 0.5^7 \cdot \frac{9!}{2!7!}$$
$$+ 0.5^3 \cdot 0.5^6 \cdot \frac{9!}{3!6!} + 0.5^4 \cdot 0.5^5 \cdot \frac{9!}{4!3!} = \sum_{i=1}^{4} 0.5^9 \frac{9!}{i!(9-i)!} = 0.5$$

- Here we compare the p-value to $\alpha/2$, as we have not taken into account that the smallest r may also be the number of signs above the median. As we have $p > \alpha/2$, we accept $H_0 : Me = 7$.

Test about the median (VI): one-sided tests

- Data: 1.34-2.45-0.33-4.73-0.68-1.02-3.13-3.45
- We want to test if median is bigger than 0.5: $H_0 : Me \geq 0.5$. $\alpha = 10\%$
- We take the median as 0.5 (boundary value),and assign signs to data: $+, +, -, +, +, +, +, + \rightarrow r^- = 1; r^+ = 7$
- We reject $H_0 : Me > 0.5$ when r^- is very big (r^+ very small).
- We look at the smallest, in order to use the tables. So in this case we have to look at the r^+ value.
- We take the critical value from the table (we have to multiply α by 2)*: $n = 8, \alpha = 0.20 \rightarrow r^* = 1$
- The value for the statistic, $r^+ = 7$, is bigger than the critical value, $r^* = 1$, so we accept that median is bigger than 0.5.

* One-sided tests with α significance levels and two-sided tests with 2α significance levels have the same critical values. As tables are given for two-sided tests, we multiply α by 2. See the annex.

Test about the median (VII): one-sided tests

A summary:

- for $H_0 : Me \leq m$, look at r^-.
- for $H_0 : Me \geq m$, look at r^+.
- Then, look for the r^* critical value on tables, after multiplying α by 2, and take the decision.

Paired samples (I): concept

- The sign test may be used to compare the medians two paired samples.
- We say that two samples are paired when they are taken over the same elements, before and after a treatment.
- E.g., we have paired samples when we measure the quantity of a substance into the blood before and after taking a drug for the same group of patients; and when we take califications over the same students at the beginning and end of the course.
- The null hypothesis may be
  $H_0 : Me(end) - M(beginning) = 0$
  (medians are the same, so if we reject that medians are just different, without giving an explicit direction);
  $H_0 : Me(end) - M(beginning) \geq 0$ (median at the end is bigger) or
  $H_0 : Me(end) - M(beginning) \leq 0$ (median at the end is smaller).
- Variable must be continuous, in order to avoid ties.

Paired samples (II): procedure

- Calculate differences for both samples values: $x(end) - x(beginning)$.
- If the difference is positive, write $+$; if negative, write $-$.
- Proceed as in the tests for one sample, taking into acconut if the test is one-sided or two-sided.

Paired samples (III): example

- In a class we have measured mathematical ability at the beginning and the end of the course:
- Beginning: 4.2-6.5-7.2-5.4-8.4-5.6-7.4-7.2
- End: 5.4-6.6-7.4-7.0-8.2-6.8-7.3-7.8
- Question: have the students bigger ability after the course? $\alpha = 0.10$
- Assign signs after calculating differences:++++-+-+
- 6 positive and 2 negative. So, we may claim that course increased the ability. Otherwise, we couldn't claim it and the test would be over.
- We take the smallest: $r = 2$.
- We take the critical values into the tables:

$$n = 8, \alpha = 0.20(2 - sided) \rightarrow r^* = 1$$

- We have $r > r^*$, so we accept that the median difference is 0, and cannot claim that the course was profitable (we should have 1 negative at most for that).

Applying the sign test: confidence interval about the median

Steps with example

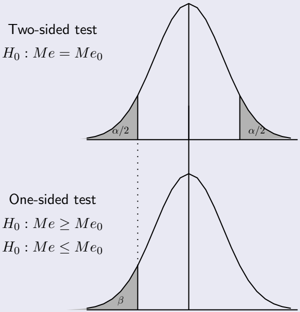
- Give 95% confidence interval about the median for these data:
- Ordered data: 6.14-6.45-8.33-11.05-5.67-5.22-7.52-10.08-12.34
- Sample size: 9.
- Give cumulated probabilities for $B(n=9, p=0.5)$ distribution.
- $(x=0, p=0.0009)$;$(x=1, p=0.01)$;$(x=2, p=0.054)$;...
- Take the value that leaves *at most* a 2.5% probability (non confidence level, 5%, divided by 2, because the interval is symmetric). At most, because we take the confidence level at least.
- That value is $x=1$. We take $x+1=2$.
- We take the $x+1=$ 2nd data fron the beginning and the end.
- So, with 98% confidence level $(1-0.01\times 2)$, we can claim that median is between 6.45 and 10.08.

Big samples

When n sample size is big, we are out of the table when we look for critical values. In such cases, we have to apply the De Moivre-Laplace theorem (in a following lesson - Normal distribution and stochastic convergence), using the normal distribution as an approximation for the binomial distribution.

Remark: we will probably solve a problem as an example for such cases in that lesson.

Annex 1: Why we take 2α in tables for one-sided sign tests



$$\beta = \alpha/2 \rightarrow \alpha = 2\beta$$