Uniform distribution

Statistics for Business

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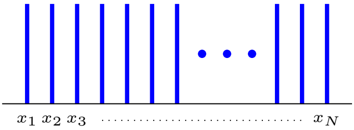
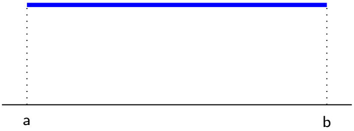
Uniform distribution

The **uniform distribution** is the probability distribution that gives the same probability to all possible values. There are two types of uniform distributions:



Discrete uniform distribution



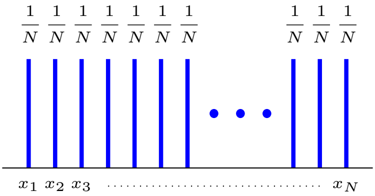
Continuous uniform distribution

We apply uniform distributions when there is absolute uncertainty about what will occur. We also use them when we take a random sample from a population, because in such cases all the elements have the same probability of being drawn. Finally, they are also usedto create random numbers, as random numbers are those that have the same probability of being drawn.

Discrete uniform distribution

Probability function

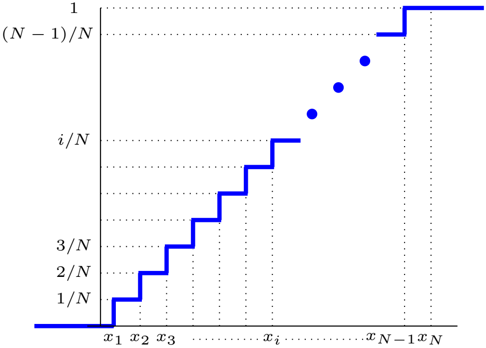
$$P[X = x] = \frac{1}{N}\ ;\ x = x_1, x_2, \cdots, x_N$$



Discrete uniform distribution

Distribution function

$$F(x) = P[X \leq x_i] = \frac{i}{N}\,;\ x_i = x_1, x_2, \cdots, x_N$$



Discrete uniform distribution

Notation, mean and variance

$$X \sim U(x_1, x_2, \ldots, x_N) \begin{cases} \mu = \dfrac{x_1 + x_N}{2} \\ \sigma^2 = \dfrac{(x_N - x_1 + 2)(x_N - x_1)}{12} \\ \quad = \dfrac{(x_N - x_1 + 1)^2 - 1}{12} \end{cases}$$

Discrete uniform distribution

Distribution of the maximum

We draw a value from a uniform distribution n times. How is distributed the maximum among those n values?

- Among n values, the maximum will be less than x_i when all of them are less than x_i:

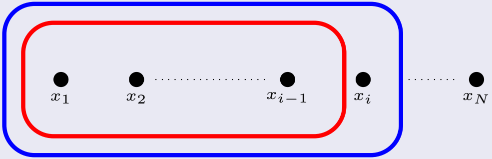
$$P[X_{max} \leq x_i] = \frac{i}{N} \times \frac{i}{N} \times \cdots \times \frac{i}{N} = \left(\frac{i}{N}\right)^n$$

- Among n values, the maximum will be less than x_{i-1} when all of them are less than x_{i-1}:

$$P[X_{max} \leq x_{i-1}] = \left(\frac{i-1}{N}\right)^n$$

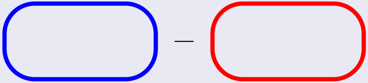
Discrete uniform distribution

Distribution of the maximum



Thus,
among n values the probability of the maximum being x_i is:

$$P[X_{max} = x_i] = \left(\frac{i}{N}\right)^n - \left(\frac{i-1}{N}\right)^n$$



Discrete uniform distribution

Distribution of the maximum: example

We throw a dice 4 times. We assume (logically) that the number of points follow a discrete uniform distribution. Calculate the probability of the maximum being 1, 2, 3, 4, 5 or 6:

- $P[X_{max} = 6] = \left(\frac{6}{6}\right)^4 - \left(\frac{5}{6}\right)^4 = 0.5177$
- $P[X_{max} = 5] = \left(\frac{5}{6}\right)^4 - \left(\frac{4}{6}\right)^4 = 0.2847$
- $P[X_{max} = 4] = \left(\frac{4}{6}\right)^4 - \left(\frac{3}{6}\right)^4 = 0.1350$
- $P[X_{max} = 3] = \left(\frac{3}{6}\right)^4 - \left(\frac{2}{6}\right)^4 = 0.0501$
- $P[X_{max} = 2] = \left(\frac{2}{6}\right)^4 - \left(\frac{1}{6}\right)^4 = 0.0115$
- $P[X_{max} = 1] = \left(\frac{1}{6}\right)^4 - \left(\frac{0}{6}\right)^4 = 0.0007$
- Probability decreases (logically)

Discrete uniform distribution

Distribution of the minimum

Likewise,

among n values the probability of the smallest being x_i is:

$$P[X_{min} = x_i] = P[X_{min} \geq x_i] - P[X_{min} \geq x_{i+1}]$$
$$= \left(\frac{N-(i-1)}{N}\right)^n - \left(\frac{N-i}{N}\right)^n$$

Discrete uniform distribution

Applications: sampling in finite populations

- When we draw a random sample from a finite population, all the elements of the population have the same probability. Thus, the model for the sampling should be the uniform discrete distribution.
- When random sampling is made with devolution, the probability of drawing a given sample of size n is:

$$P[x_1, x_2, \ldots, x_n] = \frac{1}{N} \times \frac{1}{N} \times \ldots \times \frac{1}{N} = \left(\frac{1}{N}\right)^n$$

- When random sampling is made without devolution, the probability of drawing a given sample of size n is given by the hypergeometric distribution:

$$P[x_1, x_2, \ldots, x_n] = \frac{1}{\binom{N}{n}} = \frac{1}{N} \times \frac{1}{N-1} \times \ldots \times \frac{1}{N-(n-1)} \times n!$$

Discrete uniform distribution

Application: German tank problem

- It's a classical probability problem, posed in World War II
- German tanks have a number from 1 to an unknown N number.
- Allies wanted to know that number. For that purpose, they collect the numbers of destroyed tanks.
- The model for numbers is the discrete uniform distribution: $1, 2, \ldots, N$ numbered tanks have all the same probability of being destroyed.
- Allies collect n random values from that distribution, they calculate how distributes the maximum of them and calculate the expected value, without devolution, and missing the N value:

  $$E[X_{max}] = \frac{n(N+1)}{n+1}$$

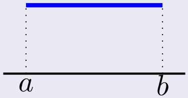
- They take the x_{max} maximum and equal to the expected value, in order to estimate the unknown N value

  $$x_{max} = \frac{n(N+1)}{n+1} \rightarrow \hat{N} = x_{max} + \frac{x_{max} - n}{n}$$

- E.g., destroyed tank numbers: 82,123,345,614.
- The estimate for the number of tanks is: $\hat{N} = 614 + \dfrac{614-4}{4} = 766.5.$

Continuous uniform distribution

Density function



$$f(x) = \frac{1}{b-a} \ ; \ a < x < b$$

$$F(x) = P[X < x] = \frac{x-a}{b-a} \ ; \ a \leq x \leq b$$

Notation, mean and variance

$$X \sim U(a,b)$$

$$X \sim U(a,b) \begin{cases} \mu = \dfrac{a+b}{2} \\[2ex] \sigma^2 = \dfrac{(b-a)^2}{12} \end{cases}$$

- The mean value is rather intuitive: as the possible values in the support have all the same probability, the mean value will in the middle.
- The wider is the interval of the support, the bigger is the dispersion, and so the variance.

Distribution of the maximum

Taken n values from a $U(a,b)$ distribution, which is the distribution of the maximum M?

- Distribution function:
$$F(M=x) = P[M<x] = \left(\frac{x-a}{b-a}\right)^n \; ; \; a \leq x \leq b$$
- Mean value: $E[M] = a + \dfrac{n(b-a)}{n+1}$
- E.g., if we want to estimate the maximum of a $U(0,10)$ distribution (that is, assuming we don't know that the maximum is 10) with 4 values, we should expect that with the maximum of those 4 values we would reach on average $0 + \frac{4\times 10}{5} = 8$, that is, 80% of the true value. With 9 data we would reach 90%.

Distribution of the minimum

Taken n values from a $U(a, b)$ distribution, which is the distribution of the minimum m?

- Distribution function:

$$F(m = x) = P[m < x] = 1 - \left(\frac{b-x}{b-a}\right)^n \;\; ; \;\; a \leq x \leq b$$

- Mean value: $E[m] = a + \dfrac{b-a}{n+1}$
- E.g., if we want to estimate the minimum of a $U(0, 10)$ distribution (that is, assuming we don't know that the minmum is 0) with 4 values, we should expect that with the minimum of those 4 values we would reach on average $0 + \frac{10}{5} = 2$. With 10 data we would get 1 on average.

Distribution of the range

Taken n values from a $U(a, b)$ distribution, which is the distribution of the range R (maximum - minimum)?

- Distribution function (for $a \leq x \leq b$):

$$F(R = x) = P[R < x] = n(\tfrac{x}{b-a})^{n-1}(\tfrac{(b-a)-x}{b-a}) + (\tfrac{x}{b-a})^{n}$$

- $E[R] = (b-a)\dfrac{n-1}{n+1}$
- E.g., if we want to estimate the range of a $U(0, 10)$ distribution (that is, assuming we don't know that the range is 10-0=10) with 4 values, we should expect that with the sample range of those 4 values we would reach on average $(10-0) \times \frac{3}{5} = 6$, 60% from the true value (10-0=10). With 10 data we would get 9/11=81% on average.

Continuous uniform distribution

Standard uniform distribution

$$X \sim U(0,1)$$

Random numbers from 0 to 1 come fron this distribution. We can create (better, simulate) them, by typing SHIFT+RAN# in the calculator.

Stochastic simulation for uniform distributions

Stochastic simulation is arficially creating data, following a given distribution.

Simulating a continuous uniform distribution is very easy compared to other distributions: random numbers follow the $U(0,1)$ distribution, and to simulate $U(a,b)$ we just have to make this linear transform: $U(a,b) = a + (b-a)U(0,1)$
So, naming sim the simulated data: $sim_{U(a,b)} = a + (b-a)sim_{U(0,1)}$